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SPEKKENS'S SYMMETRIC NO-GO THEOREM

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In [6], Spekkens clarifies the ways in which classical theories differ from quantum mechanics. He improves the traditional notions of non-negativity of Wigner-style quasi-probability distributions and non-contextuality of observations. He argues that the improvements more accurately capture what a classical universe would look like. Thus, both of these improved notions serve to distinguish quantum theory from classical theories, in particular from theories that use hidden variables in an attempt to explain the results of quantum mechanics on a classical basis. Spekkens then shows that the two improved notions are equivalent to each other.

Spekkens's improvements of non-negativity and non-contextuality emphasize the involvement of both preparations and measurements. In the second part of [6], Spekkens provides what he calls an even-handed approach to a no-go theorem. The theorem asserts that the requirement of non-contextuality (or equivalently of non-negativity) prevents a theory from matching the predictions of quantum mechanics; in other words, non-contextual hidden-variable theories can't succeed. "Even-handed" means that the proof treats preparations and measurements in a symmetrical way.

The paper [6] contains some minor inaccuracies and one false claim, in the proof of the no-go theorem. The false claim is "that a function f that is convex-linear on a convex set S of operators that span the space of Hermitian operators (and that takes value zero on the zero operator if the latter is in S) can be uniquely extended to a linear function on this space." Unfortunately, this claim, early in the proof, is used in an essential way in the rest of the argument. In this note, we analyze carefully Spekkens's proof of the no-go theorem, explain the inaccuracies, reduce the task of proving the no-go theorem to the special case of a single qubit and then prove the special case. This gives us a complete proof of Spekkens's no-go theorem. An alternative proof of the no-go theorem is given in the series of papers [1, 2, 3].

1. Definitions

Spekkens defines a *quasiprobability representation* of a quantum system by the following features.

QPR1 Every density operator ρ is represented by a normalized and real-valued function μ_ρ on a measurable space Λ.

QPR2 Every positive operator-valued measure (POVM) $\{E_k\}$ is represented by a set $\{\xi_{E_k}\}$ of real-valued functions on Λ that sum to the unit function on Λ. (The trivial POVM $\{I\}$ is represented by $\xi_I(\lambda) = 1$, and the zero operator is represented by the zero function.)

QPR3 For all density operators ρ and all POVM elements E_k, we have $\mathrm{Tr}(\rho E_k) = \int d\lambda\, \mu_\rho(\lambda)\xi_{E_k}(\lambda)$.

A quasiprobability representation is called *nonnegative* if all the functions μ_ρ and ξ_E take only nonnegative values.

We begin our analysis by looking carefully at the notions used in this definition of quasiprobability representation and clarifying some aspects of the definition.

1.1. Density operator.

Within the definition of quasiprobability representation, Spekkens explains "density operator" as "a positive trace-class operator on a Hilbert space $\mathcal{H}$". Although "trace-class" implies that the operator ρ has a well-defined trace, Spekkens presumably intended more, namely that the trace $\mathrm{Tr}(\rho)$ should be equal to 1. This would conform with the usual meaning of "density operator"; it would also account for the requirement that μ_ρ be normalized. If one could multiply ρ by a positive real factor and still have a density operator, then the associated μ_ρ should also be multiplied by the same factor. From now on, we shall assume that "trace 1" is included in the definition of density operator.

It is also worth remembering that "trace-class" is important only in the context of infinite-dimensional Hilbert spaces. If $\mathcal{H}$ is finite-dimensional, then all (linear) operators on it are in the trace class. Spekkens's proof of the no-go theorem does not require an infinite-dimensional space; it works as long as the dimension of $\mathcal{H}$ is at least 2. So for many purposes, we need not worry about the "trace-class" clause in the definition of density operators.

1.2. Measurable space.

The phrase "measurable space" is standard terminology for a set X together with a σ-algebra Σ of subsets of X, the members of Σ being called the measurable sets. A measurable space differs from a measure space in that the latter has, in addition to

X and Σ, a countably additive measure defined on all the measurable sets.

We believe that Spekkens intends Λ to be not merely a measurable space but a measure space. He uses the formula $\int \mu_\rho(\lambda)\, d\lambda = 1$ as the definition of the requirement in QPR1 that μ_ρ be normalized. This integral and the one in clause QPR3 of the definition of quasiprobability representation both presuppose the presence of a measure to make sense of $d\lambda$. They also presuppose that the functions μ_ρ are measurable.

An alternative modification to make sense of these integrals would be to change the requirement that ρ is represented by a *function* and to require instead that it be represented by a *measure*, say ν_ρ. The notation $\mu_\rho(\lambda)\, d\lambda$ could then be taken to be syntactic sugar for $d\nu_\rho(\lambda)$. This alternative approach has, as far as we can see, two disadvantages and two advantages. The first disadvantage is that it requires us to understand Spekkens's notation $\mu_\rho(\lambda)\, d\lambda$, which looks like a standard notation, as syntactic sugar for something rather different. The second is that it explicitly contradicts Spekkens' assertion that μ_ρ should be a function. The first advantage is that it preserves Spekkens's convention that Λ is merely a measurable space, not a measure space. The second advantage is that it is more general. In the approach with an *a priori* given measure $d\lambda$, multiplying it by functions μ_ρ produces only those measures $\mu_\rho(\lambda)\, d\lambda$ that are absolutely continuous with respect to $d\lambda$. The alternative approach allows arbitrary measures (on the given σ-algebra Σ) without any requirement of absolute continuity.

1.3. **Positive operator-valued measures.** The second defining feature, QPR2, of a quasiprobability representation represents positive operator valued measures[1] $\{E_k\}$ by sets of functions ξ_{E_k}. The elements E_k of a POVM are positive Hermitian operators such that $I - E_k$ is also positive. That is, the spectrum of E_k lies in the interval $[0, 1]$ of the real line. Conversely, any such operator occurs as a member of some POVM, and usually as a member of many POVMs. Specifically, if E is a positive Hermitian operator and $I - E$ is also positive, then $\{E, I - E\}$ is a POVM; unless $E = I$, we can replace $I - E$ in this POVM by two or more positive operators whose sum is $I - E$, thereby obtaining other POVMs containing E.

[1]We follow Spekkens's usage of "POVM" to refer to a discrete set of operators. This usage agrees with the standard text [4]. There is a generalization, involving operator-valued measures; see for example [8]. For our purposes, the simpler version is adequate, since the no-go theorem for these simpler POVMs implies the theorem for the broader class.

The question arises whether the function ξ_{E_k} in a quasiprobability representation can depend on the POVM from which E_k was taken or must depend only on the operator E_k itself. The wording of the definition suggests the former, while the notation ξ_{E_k} suggests the latter. Fortunately for our purposes, Spekkens's definition of "measurement noncontextuality" requires that ξ_{E_k} "depends only on the associated POVM *element* E_k" (italics added). Since our goal in this paper, the no-go theorem, is about noncontextual representations, we can safely follow the notation and assume that ξ_E depends only on E, not on the POVM in which it occurs (and, a fortiori, not on the measurement process by which that POVM is realized).

2. An additional hypothesis

At the beginning of his proof of the no-go theorem, Spekkens notes that a mixture $\rho = \sum_j w_j \rho_j$ of density operators ρ_j with weights w_j can be prepared by first randomly choosing one value of j from the probability distribution $\{w_j\}$ and then preparing ρ_j. He infers that "clearly" $\mu_\rho(\lambda) = \sum_j w_j \mu_{\rho_j}(\lambda)$.

Although this inference is highly plausible and natural on physical grounds, it does not follow from just the definition of quasiprobability distribution as quoted above. Suppose that the functions ξ_E do not span the whole space of square-integrable functions on Λ, so that there is a function σ orthogonal to all of these ξ_E's, where "orthogonal" means that $\int \sigma(\lambda)\xi_E(\lambda)\, d\lambda = 0$. One could modify the μ_ρ functions by adding to each one some multiple of σ, obtaining $\mu'_\rho = \mu_\rho + c_\rho \sigma$ and still satisfying the definition of quasiprobability representation. Here the coefficients c_ρ can be chosen arbitrarily for each density operator ρ. By choosing them in a sufficiently incoherent way, one could arrange that $\mu'_\rho(\lambda) \neq \sum_j w_j \mu'_{\rho_j}(\lambda)$.

If, on the other hand, the ξ_E's do span the whole space of functions on Λ, then Spekkens's desired equation $\mu_\rho(\lambda) = \sum_j w_j \mu_{\rho_j}(\lambda)$ does follow, for all but a measure-zero set of λ's, because the two sides of the equation must give the same result when integrated against any ξ_E.

Unfortunately, nothing in the definition of quasiprobability representations requires the ξ_E's to span the whole space. For example, given any quasiprobability representation, we can obtain another, physically equivalent one as follows. Replace Λ by the disjoint union $\Lambda_1 \sqcup \Lambda_2$ of two copies of Λ. Define the measure of any subset of $\Lambda_1 \sqcup \Lambda_2$ to be the average of the original measures of its intersections with the two copies of Λ. Define all the functions μ_ρ and ξ_E on the new space by

simply copying the original values on both of the Λ_i's. The result is a quasiprobability representation in which the ξ_E's span only the space of functions that are the same on the two copies of Λ.

The result of this discussion is that, in order to prove the no-go theorem along the lines proposed by Spekkens, we must add an additional hypothesis about mixtures of densities. There is a similar assumption for mixtures of measurements.

Convex-linearity Hypothesis: Let $\{w_j\}$ be a probability distribution on a set of indices j.

- If $\rho = \sum_j w_j \rho_j$, then $\mu_\rho(\lambda) = \sum_j w_j \mu_{\rho_j}(\lambda)$.
- If $E = \sum_j w_j E_j$, then $\xi_E(\lambda) = \sum_j w_j \xi_{E_j}(\lambda)$.

This hypothesis is exactly statements (7) and (8) in [6]. The name of the hypothesis refers to the following terminology, which we shall need again later.

Definition 1. Let C be a convex set in a real vector space V, and let f be a function from C into another real vector space W. Then f is *convex-linear* on a subset S of C if

$$f(a_1 v_1 + \cdots + a_n v_n) = a_1 f(v_1) + \cdots + a_n f(v_n)$$

for all vectors $v_1, \ldots, v_n \in S$ and all nonnegative numbers $a_1, \ldots, a_n$ with $a_1 + \cdots + a_n = 1$.

Thus, the convex-linearity hypothesis says that the functions $\rho \mapsto \mu_\rho$ and $E \mapsto \xi_E$ are convex-linear on the sets of density matrices and POVM elements, respectively.

3. The no-go theorem

On an intuitive level, the no-go theorem asserts that nonnegative quasiprobability representations[2] subject to the convex-linearity hypothesis cannot reproduce the predictions of quantum mechanics. A considerable amount of agreement with quantum mechanics is already built into the definition of quasiprobability representations. Specifically, the equation $\mathrm{Tr}(\rho E_k) = \int \mu_\rho(\lambda) \xi_{E_k}(\lambda)\, d\lambda$ says that the expectation of E_k in state ρ is the same whether computed by the quantum formula $\mathrm{Tr}(\rho E_k)$ or as an average using the functions μ_ρ and ξ_{E_k} from the quasiprobability representation. Spekkens's no-go theorem asserts that there is no nonnegative quasiprobability representation satisfying convex-linearity.

[2]These are equivalent to noncontextual ontological models, as Spekkens shows in the earlier sections of [6].

A small technical point is that the no-go theorem presupposes that the quantum mechanics is non-trivial. Quantum mechanics on Hilbert spaces of dimensions 0 or 1 is classical (and trivial), so we must assume that we are dealing with a Hilbert space $\mathcal{H}$ of dimension at least 2. An inspection of Spekkens's argument reveals that he never uses any stronger assumptions about $\mathcal{H}$. Thus, the no-go theorem can be formally stated as follows.

Theorem 2. *For a Hilbert space $\mathcal{H}$ of dimension at least two, there is no way to define nonnegative μ_ρ, for all density operators ρ, and to define nonnegative ξ_E, for all positive Hermitian operators E with $I - E$ positive, so as to satisfy both the definition of a quasiprobability representation and the convex-linearity hypothesis.*

4. Reduction to two dimensions

In this section, we reduce the task of proving Spekkens's no-go theorem to the special case where $\mathcal{H}$ has dimension 2. (In the terminology of quantum computing, $\mathcal{H}$ represents a single qubit.) More generally, we show that, if there were a nonnegative quasiprobability representation satisfying convex-linearity for some Hilbert space $\mathcal{H}$, then there would also be such a representation, using the same measure space Λ, for any nonzero, closed subspace $\mathcal{H}'$ of $\mathcal{H}$.

To see this, suppose functions μ_ρ (for all ρ) and ξ_E (for all E) constitute such a representation for $\mathcal{H}$. Let $i : \mathcal{H}' \to \mathcal{H}$ be the inclusion map (the identity map of $\mathcal{H}'$ regarded as a map into $\mathcal{H}$), and let $p : \mathcal{H} \to \mathcal{H}'$ be the orthogonal projection map (sending each vector in $\mathcal{H}'$ to itself and sending each vector orthogonal to $\mathcal{H}'$ to 0). Also, fix some unit vector $|\alpha\rangle \in \mathcal{H}'$.

Each density operator ρ on $\mathcal{H}'$ gives rise to a density operator $\bar{\rho} = i \circ \rho \circ p$ on $\mathcal{H}$. For pure states, this amounts to just considering a state vector in $\mathcal{H}'$ as a vector in the larger Hilbert space $\mathcal{H}$. For mixed states, the extension preserves averages. We begin defining a quasiprobability representation for $\mathcal{H}'$ by setting $\mu'_\rho = \mu_{\bar{\rho}}$. We note that this is a normalized nonnegative real-valued function on Λ, and that it satisfies the part of convex-linearity that refers to the representations of densities.

It is tempting to proceed exactly analogously with POVM elements E and their representing functions ξ_E. That procedure doesn't quite work, because the definition of quasiprobability representation imposes a specific requirement on ξ_I, where I is the identity operator. Unfortunately, if I is the identity operator on $\mathcal{H}'$, then $i \circ I \circ p$ is not the identity operator on $\mathcal{H}$. So we must proceed slightly differently, and it is here that the fixed unit vector $|\alpha\rangle$ will be useful.

Given a POVM element E on $\mathcal{H}'$, i.e., a positive, Hermitian operator such that $I - E$ is also positive, we define $\bar{E}$ to be the unique linear operator on $\mathcal{H}$ such that

$$\bar{E}|\psi\rangle = \begin{cases} E|\psi\rangle & \text{if } \psi \in \mathcal{H}', \\ \langle\alpha|E|\alpha\rangle|\psi\rangle & \text{if } |\psi\rangle \perp \mathcal{H}'. \end{cases}$$

In other words, $\bar{E}$ agrees with E on $\mathcal{H}'$ and with a scalar multiple of the identity on the orthogonal complement of $\mathcal{H}'$, the multiplier of the identity being $\langle\alpha|E|\alpha\rangle$. This extension process produces POVM elements for $\mathcal{H}$; indeed, if a set $\{E_k\}$ of operators is a POVM for $\mathcal{H}'$, then $\{\bar{E}_k\}$ is a POVM for $\mathcal{H}$. Furthermore, the extension process sends the identity and zero operators on $\mathcal{H}'$ to the identity and zero operators on $\mathcal{H}$, and the process respects weighted averages.

We continue the definition of a quasiprobability representation for $\mathcal{H}'$ by setting $\xi'_E = \xi_{\bar{E}}$ for all POVM elements E on $\mathcal{H}'$. The remarks above immediately imply that these functions ξ'_E are as required by the second part, QPR2, of the definition of quasiprobability representation, that they are nonnegative, and that they satisfy the relevant part of the convex-linearity hypothesis.

To verify the last part, QPR3, of the definition of quasiprobability representation, we observe that, for any density operator ρ and POVM element E on $\mathcal{H}'$, the extensions $\bar{\rho}$ and $\bar{E}$ agree with ρ and E on $\mathcal{H}'$, while on the orthogonal complement of $\mathcal{H}'$, $\bar{\rho}$ acts as zero and $\bar{E}$ acts as a scalar multiple of the identity. It follows immediately that $\mathrm{Tr}(\bar{\rho}\bar{E}) = \mathrm{Tr}(\rho E)$, and therefore

$$\mathrm{Tr}(\rho E) = \int d\lambda\, \mu'_\rho(\lambda)\xi'_E(\lambda),$$

as required.

This completes the proof that nonnegative quasiprobability representations subject to convex-linearity can be "restricted" to nonzero, closed subspaces of the original Hilbert space. Therefore, it suffices to prove the no-go theorem in the special case where $\mathcal{H}$ has dimension 2.

Remark 3. By concentrating on the case of dimension 2, we gain two advantages. First, we can avoid some technicalities that would arise for infinite-dimensional Hilbert spaces. Second, we obtain a more concrete picture of the relevant spaces of density operators and measurements. (The first of these advantages would result from reduction to any finite number of dimensions; the second benefits specifically from dimension 2.)

5. Convex-linear transformations

Spekkens asserts that, if a function f is convex-linear on a convex set $\mathcal{S}$ of operators that span the space of Hermitian operators (and f takes the value zero on the zero operator if the latter is in $\mathcal{S}$), then f can be uniquely extended to a linear function on this space. Unfortunately, such a linear extension need not exist in the general case, when zero is not in $\mathcal{S}$.[3] For a simple example, consider the function that is identically 1 on an $\mathcal{S}$ that spans the space of Hermitian operators, does not contain 0, but does contain two orthogonal projections and their sum.

The correct version of the result extends f not to a linear function but to translated-linear function, i.e., a composition of translations and a linear function. The rest of this section is devoted to a proof of this fact, in somewhat greater generality than we need. It applies to arbitrary real vector spaces; that the space consists of Hermitian operators is irrelevant.

The *convex hull*, $\mathrm{Conv}(S)$, of a subset S of a real vector space V consists of the convex combinations $a_1v_1 + \cdots + a_nv_n$ of vectors $v_1, \ldots, v_n \in S$ where $a_1 + \cdots + a_n = 1$ and every $a_i \geq 0$. The *affine hull*, $\mathrm{Aff}(S)$, of S consists of the affine combinations $a_1v_1 + \cdots + a_nv_n$ of vectors $v_1, \ldots, v_n \in S$ where $a_1 + \cdots + a_n = 1$ but some coefficients a_i may be negative.

A set is *convex* if it contains all the convex combinations of its members; similarly, it is an *affine* space if it contains all the affine combinations of its members. An easy computation shows that convex hulls are convex and affine hulls are affine spaces; that is $\mathrm{Conv}(\mathrm{Conv}(S)) = \mathrm{Conv}(S)$ and $\mathrm{Aff}(\mathrm{Aff}(S)) = \mathrm{Aff}(S)$.

An affine space A in a vector space V is said to be *parallel* to a linear subspace L of V if $A = u_0 + L = \{u_0 + v : v \in L\}$ for some $u_0 \in V$. It is easy to see that, if an affine space A is parallel to a linear space L as above, then (i) L is unique, (ii) $u_0 \in A$, (iii) any vector in A can play the role of the translator u_0, and (iv) A is either equal to L or disjoint from L.

[3]Spekkens gives a formula purporting to define a linear extension of f in general, but it is not well-defined because it involves some arbitrary choices. He also gives, in footnote 18 of the newer version [7] of his paper, an argument purporting to show that his formula is independent of those choices, but that argument fails. It involves dividing by an appropriate constant C to turn two nonnegative linear combinations, the two sides of an equation, into convex combinations so that the assumption of convex-linearity can be applied. But the necessary divisor C may need to be different for the two sides of the equation.

Lemma 4 (§1 in [5])**.** *Any affine subspace A of a real vector space V is parallel to a linear subspace L of V.*

In other words, any affine subspace is a translation of a linear subspace. For example, in $\mathbb{R}^2$, we have that $\mathrm{Aff}\{(0,1),(1,0)\}$ is parallel to the diagonal $y=-x$, and $\mathrm{Aff}\{(0,1),(1,0),(1,1)\}$ is (and thus is parallel to) $\mathbb{R}^2$.

Proof. If A contains the zero vector $\vec{0}$ then it is a linear subspace. Indeed, if $v \in A$ then any multiple $av = av + (1-a)\vec{0} \in A$. And if $u, v \in A$ then $u + v = 2(\frac{1}{2}u + \frac{1}{2}v) \in A$.

For the general case, let u_0 be any vector in the affine space A. It suffices to show that $L = \{v - u_0 : v \in A\}$ is an affine space, because then the preceding paragraph shows that it is a linear space, and clearly $A = u_0 + L$. Any affine combination $a_1(v_1 - u_0) + \cdots + a_n(v_n - u_0)$ of vectors in L (so the v_i are in A and the sum of the a_i is 1) can be rewritten as $(a_1v_1 + \cdots + a_nv_n) - u_0$, which is in L. □

Let V and W be real vector spaces, S a subset of V, $C = \mathrm{Conv}(S)$ its convex hull, and $A = \mathrm{Aff}(S)$ its affine hull. Recall that a transformation $f : C \to W$ is convex-linear on S if

$$f(a_1v_1 + \cdots + a_nv_n) = a_1f(v_1) + \cdots + a_nf(v_n)$$

for any convex combination $a_1v_1 + \cdots + a_nv_n$ of vectors v_i from S. A transformation $f : A \to W$ is *translated-linear* if it has the form $f(v) = w_0 + h(v - u_0)$ for some $w_0 \in W$, some $u_0 \in A$, and some linear function $h : L \to W$ defined on the linear space $L = A - u_0$ parallel to A.

Proposition 5. *With notation as above, any transformation $f : C \to W$ that is convex-linear on S has a unique extension to a translated-linear function on A.*

Proof. Notice first that translations $v \mapsto v - u_0$ and linear functions both preserve affine combinations. A translated-linear function, being the composition of two translations and a linear function, therefore also preserves affine combinations. This observation implies the uniqueness part of the proposition. Indeed, every element of A is an affine combination $a_1s_1 + \cdots + a_ns_n$ of elements of S, and therefore any translated-linear extension of f must map it to $a_1f(s_1) + \cdots + a_nf(s_n)$.

To prove the existence part of the proposition, it will be useful to work with the graphs of functions. For any function $g : S \to W$ with $S \subseteq V$, its *graph* is the subset of $V \oplus W$ consisting of the pairs $(s, g(s))$

for $s \in S$.[4] We record for future reference that the graph of g is a linear subspace of $V \oplus W$ if and only if the domain of g is a linear subspace of V and g is a linear transformation from that domain to W. We also note that the projection $\pi : V \oplus W \to V : (v, w) \mapsto v$ is a linear transformation that sends the graph of any g to the domain of g.

In the situation of the proposition, let $f : C \to W$ be a transformation that is convex-linear on S, and let $F \subseteq V \oplus W$ be its graph. Also, let F^- be the graph of the restriction of f to S. Notice that the convex-linearity of f on S means exactly that F is the convex hull of F^-. It follows that F and F^- have the same affine hull, because

$$\mathrm{Aff}(F) = \mathrm{Aff}(\mathrm{Conv}(F^-)) \subseteq \mathrm{Aff}(\mathrm{Aff}(F^-)) = \mathrm{Aff}(F^-) \subseteq \mathrm{Aff}(F).$$

We claim that this affine hull $\mathrm{Aff}(F^-)$ is the graph of a function; that is, it does not contain two distinct elements (v, w) and (v, w') with the same first component v. To see this, suppose we had two such elements in $\mathrm{Aff}(F) = \mathrm{Aff}(F^-)$, say

$$(v, w) = a_1(s_1, f(s_1)) + \cdots + a_m(s_m, f(s_m))$$

and

$$(v, w') = b_1(t_1, f(t_1)) + \cdots + b_n(t_n, f(t_n)),$$

where all the s_i's and t_j's are in S and where

$$a_1 + \cdots + a_m = b_1 + \cdots + b_n, \tag{1}$$

because both sides are equal to 1. So we have

$$a_1 s_1 + \cdots + a_m s_m = b_1 t_1 + \cdots + b_n t_n, \tag{2}$$

because both sides are equal to v, and we want to prove $w = w'$, i.e.,

$$a_1 f(s_1) + \cdots + a_m f(s_m) = b_1 f(t_1) + \cdots + b_n f(t_n). \tag{3}$$

In the special case where all coefficients a_i and b_j are ≥ 0, vector v is in C and both sides of (3) are equal to $f(v)$. The general case reduces to this special case as follows. In all three equations (1)–(3), move every summand with a negative coefficient to the other side, and then divide the resulting equations by the left part of the rearranged equation (1). As a result we return to the special case already treated. Since the old version of (3) follows from the new one, this completes the proof of our claim that $\mathrm{Aff}(F) = \mathrm{Aff}(F^-)$ is the graph of a function.

By Lemma 4, the affine space $\mathrm{Aff}(F)$ is parallel to a linear subspace H of $V \oplus W$, say $\mathrm{Aff}(F) = (u_0, w_0) + H$, where $u_0 \in V$ and $w_0 \in W$. From the fact that $\mathrm{Aff}(F)$ is the graph of a function, it follows immediately that H is also the graph of a function. Indeed, if H contains (v, w) and

[4] In set-theoretic foundations, a function is usually defined as a set of ordered pairs, and so g is the same thing as its graph.

(v, w'), then $\mathrm{Aff}(F)$ contains $(v - u_0, w - w_0)$ and $(v - u_0, w' - w_0)$, so $w - w_0 = w' - w_0$ and $w = w'$.

Let h be the function whose graph is H. Because H is a linear subspace of $V \oplus W$, we know that h is a linear transformation from some linear subspace L of V into W.

The fact that $(u_0, w_0) + H = \mathrm{Aff}(F)$ tells us, by applying the linear projection $\pi : V \oplus W \to V$, that $u_0 + L$ equals

$$\pi(\mathrm{Aff}(F)) = \mathrm{Aff}(\pi(F)) = \mathrm{Aff}(C) = A,$$

where the first equality comes from linearity of π and the second from the fact that F is the graph of the function f whose domain is C. So A is parallel to the linear subspace L of V. Furthermore, for each $v \in C$, we have

$$(v, f(v)) \in F \subseteq \mathrm{Aff}(F) = (u_0, w_0) + H,$$

so $(v - u_0, f(v) - w_0)$ is in the graph H of h. That is, $h(v - u_0) = f(v) - w_0$ and so $f(v) = w_0 + h(v - u_0)$. Thus, the translated-linear function $v \mapsto w_0 + h(v - u_0)$ is the desired extension of f. □

Remark 6. A linear function h on a subspace L of a vector space V can be extended to a linear function $\bar{h}$ on all of V. Extend any basis of L to a basis of V, define $\bar{h}$ arbitrarily on the new basis vectors that are not in L, and extend the resulting function by linearity to all of V.

For transformations defined on all of V, we have a simpler formula for translated-linear functions, because

$$w_0 + \bar{h}(v - u_0) = w_0 + \bar{h}(v) - \bar{h}(u_0) = \bar{h}(v) + w_1,$$

where $w_1 = w_0 - \bar{h}(u_0)$.

On the other hand, in contrast to Proposition 5, this $\bar{h}$ is not unique (unless $L = V$).

Also, in the case of infinite-dimensional spaces, the extension process requires the axiom of choice (to extend bases) and need not be well-behaved with respect to natural topologies on the vector spaces.

6. Density operators and POVM elements in two dimensions

In this section, we recall the form of density operators and POVM elements in the case where $\mathcal{H}$ is two-dimensional. In this case, a basis for the Hermitian operators on $\mathcal{H}$ is given by the identity and the three Pauli matrices

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

It will be convenient to use vector notation, denoting the triple of matrices (X, Y, Z) by $\vec{X}$. Then the general Hermitian matrix looks like

$$wI + xX + yY + zZ = wI + \vec{x} \cdot \vec{X},$$

where w and the three components of $\vec{x}$ are real numbers. The eigenvalues of this Hermitian matrix are

$$w \pm \sqrt{x^2 + y^2 + z^2} = w \pm \|\vec{x}\|$$

In particular, the trace of this matrix is $2w$, and the matrix is positive if and only if $w \geq \|\vec{x}\|$.

Density matrices are the Hermitian, positive matrices of trace 1, so they have the form

$$\rho = \rho(\vec{x}) = \frac{1}{2}(I + \vec{x} \cdot \vec{X}),$$

where $\|\vec{x}\| \leq 1$. As indicated by the notation, we parametrize these density matrices by three-component vectors $\vec{x}$ of norm ≤ 1. The three-dimensional ball that serves as the parameter space here is called the Bloch sphere (with its interior).

Similarly, POVM elements have the form

$$E = E(m, \vec{p}) = mI + pX + qY + rZ = mI + \vec{p} \cdot \vec{X}$$

with

$$\|\vec{p}\| \leq m \leq 1 - \|\vec{p}\|$$

(because E and $I - E$ are positive operators) and therefore $\|\vec{p}\| \leq \frac{1}{2}$. The parameter space here, consisting of all four-component vectors satisfying these inequalities, is a double cone over a three-dimensional ball of radius $\frac{1}{2}$.

We record for future reference the traces

$$\mathrm{Tr}(I) = 2, \quad \mathrm{Tr}(X) = \mathrm{Tr}(Y) = \mathrm{Tr}(Z) = 0$$

and the multiplication table

$$XY = -YX = iZ, \quad YZ = -ZY = iX, \quad ZX = -XZ = iY,$$

and

$$X^2 = Y^2 = Z^2 = I.$$

From these facts, it is easy to compute that

$$\mathrm{Tr}(\rho(\vec{x})E(m, \vec{p})) = m + \vec{x} \cdot \vec{p},$$

where the factor $\frac{1}{2}$ in the definition of $\rho(\vec{x})$ has cancelled the factor 2 arising from $\mathrm{Tr}(I)$.

7. Quasiprobability representation

Finally, we are ready to prove Theorem 2.

Suppose, toward a contradiction, that we have a nonnegative quasiprobability representation satisfying convex-linearity, for a two-dimensional $\mathcal{H}$. In view of Proposition 5, we know that

$$\mu_{\rho(\vec{x})}(\lambda) = \vec{x} \cdot \vec{A}(\lambda) + C(\lambda)$$

and

$$\xi_{E(m,\vec{p})} = \vec{p} \cdot \vec{B}(\lambda) + mD(\lambda) + F(\lambda)$$

for some nine functions $A_i(\lambda), B_i(\lambda), C(\lambda), D(\lambda), F(\lambda)$ where the index i ranges from 1 to 3. (The "translated" part of "translated-linear" accounts for C and F.)

The definition of quasiprobability representation leads to some simplifications. $E(0, \vec{0})$ is the zero operator, whose associated ξ function is required to be identically zero. That gives us $F(\lambda) = 0$ for all λ, so we can simply omit F from the formula for ξ.

Also, $E(1, \vec{0})$ is the identity operator, whose associated ξ function is required to be identically 1. That gives us $D(\lambda) = 1$ for all λ. So we can simplify the ξ formula above to read

$$\xi_{E(m,\vec{p})} = \vec{p} \cdot \vec{B}(\lambda) + m.$$

Next, consider the requirement that

$$\mathrm{Tr}(\rho(\vec{x})E(m, \vec{p})) = \int \xi_{E(m,\vec{p})}\mu_{\rho(\vec{x})}\, d\lambda.$$

We already evaluated the trace on the left side of this equation at the end of the preceding section. The integral on the right side is

$$\int [(\vec{p} \cdot \vec{B}(\lambda))(\vec{x} \cdot \vec{A}(\lambda)) + (\vec{p} \cdot \vec{B}(\lambda))C(\lambda) + m(\vec{x} \cdot \vec{A}(\lambda)) + mC(\lambda)]\, d\lambda.$$

Comparing the trace and the integral, and equating coefficients of the various monomials in m, $\vec{p}$, and $\vec{x}$, we find that

$$\int B_i(\lambda)A_j(\lambda)\, d\lambda = \delta_{i,j}, \tag{4}$$

$$\int B_i(\lambda)C(\lambda)\, d\lambda = 0, \tag{5}$$

$$\int A_i(\lambda)\, d\lambda = 0, \text{and} \tag{6}$$

$$\int C(\lambda)\, d\lambda = 1. \tag{7}$$

Next, we extract as much information as we can from the assumption that all the functions μ_ρ and ξ_E are nonnegative.

In the case of ξ_E, this means that, as long as $\|\vec{p}\| \leq m, 1-m$ (so that $E(m,\vec{p})$ is a POVM element), we must have $m + \vec{p}\cdot\vec{B}(\lambda) \geq 0$ for all λ. Temporarily consider a fixed λ and a fixed $m \in [0, \frac{1}{2}]$. To get the most information out of the inequality $m + \vec{p}\cdot\vec{B}(\lambda) \geq 0$, we choose the "worst" vector $\vec{p}$, i.e., we make $\vec{p}\cdot\vec{B}(\lambda)$ as negative as possible, by choosing $\vec{p}$ in the opposite direction to $\vec{B}(\lambda)$ and with the largest permitted magnitude, namely m. That is, we take

$$\vec{p} = -\frac{m}{\|\vec{B}(\lambda)\|}\vec{B}(\lambda)$$

so that our inequality becomes $0 \leq m(1 - \|\vec{B}(\lambda)\|)$, and therefore

$$\|\vec{B}(\lambda)\| \leq 1 \qquad \text{for all } \lambda.$$

Repeating the exercise for $m \in [\frac{1}{2}, 1]$ gives no new information.

So we turn to the case of $\mu_{\rho(\vec{x})}$, for which the nonnegativity requirement reads

$$\vec{x}\cdot\vec{A}(\lambda) + C(\lambda) \geq 0.$$

For each fixed λ, we consider the "worst" $\vec{x}$, namely a vector $\vec{x}$ in the direction opposite to $\vec{A}(\lambda)$ and with the maximum allowed magnitude, namely 1. So we take

$$\vec{x} = -\frac{\vec{A}(\lambda)}{\|\vec{A}(\lambda)\|}$$

and obtain the inequality $0 \leq -\|\vec{A}(\lambda\| + C(\lambda)$. Thus, we have

$$\|\vec{A}(\lambda)\| \leq C(\lambda) \qquad \text{for all } \lambda.$$

In particular, $C(\lambda)$ is everywhere nonnegative.

A trivial consequence of $\|\vec{A}(\lambda)\| \leq C(\lambda)$ is that $|A_1(\lambda)| \leq C(\lambda)$. Similarly, a trivial consequence of $\|\vec{B}(\lambda)\| \leq 1$ is $|B_1(\lambda)| \leq 1$. Putting this information into the $i = j = 1$ case of equation (4), and also using (7), we find that

$$1 = \left|\int B_1(\lambda)A_1(\lambda)\,d\lambda\right| \leq \int |B_1(\lambda)|\cdot|A_1(\lambda)|\,d\lambda \leq \int 1\cdot C(\lambda)\,d\lambda = 1.$$

So both of the inequalities here must be equalities. In particular, $|B_1(\lambda)| = 1$ for almost all λ except where $C(\lambda) = 0$.

Similarly, we get that, for almost all λ except where $C(\lambda) = 0$, we also have $|B_2(\lambda)| = |B_3(\lambda)| = 1$ and therefore $\|\vec{B}(\lambda)\| = \sqrt{3}$. Since we also know $\|\vec{B}(\lambda)\| \leq 1$, we must conclude that $C(\lambda) = 0$ almost

everywhere. But that contradicts equation (7), and so the proof of the no-go theorem is complete.

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